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The existence of contact transformations for evolution-type equations

E Momoniat and F M Mahomed

Centre for Differential Equations, Continuum Mechanics and Applications, Department of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

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Abstract. We investigate the existence of a one-parameter group of contact transformations for evolution-type equations $u_t = F(t, x, u, u_x, u_{xx}, \dots, u_{(n)})$ (subscripts denote differentiation unless otherwise indicated), where $u_{(n)}$ is the nth derivative of u with respect to x. We prove that contact transformations of evolution equations, where F is expandable as a power series in terms of all derivatives of order higher than one, are just extended Lie point transformations. This result is extended to the case with m independent space variables. As a consequence, we obtain an ansatz for determining Lie point transformations for nth-order evolution equations with m independent space variables. Examples are given to verify the results obtained as well as to show how Lie point transformations of these evolution-type partial differential equations can be calculated from this ansatz.

1. Introduction

Evolution equations model a wide variety of phenomena in the physical, biological and economic sciences. These phenomena range in diversity from heat conduction [1, 2], diffusion of particles within a media [3–5], stock option pricing on financial exchanges [6] and the study of waves in quantum mechanics [7, 8]. Solving these evolution (diffusion) equations may not be trivial, especially if they are nonlinear or have a dependence on arbitrary functions. Lie group theory provides a useful tool for the solution of these partial differential equations. Many books have been written on this aspect [9–13]. For Lie group theory to be useful for the solution of evolution-type partial differential equations, the Lie point transformation generators need to be determined [9–13]. Once the Lie point transformation generators have been determined, they can be used to obtain special solutions (group-invariant solutions) of the differential equations under consideration. A reduction in the number of variables and transformations to other simpler equations which may be easier to solve are also possible. Lie point transformation generators and their applications to some evolution equations are listed in [14]. Lie theory has provided insight into many physical phenomena, which may otherwise not have been possible. In [1] a general similarity solution for the heat equation is determined. In [8], possible forms of the interaction term F of the time-dependent Schrödinger equation $u_{xx} + \mathrm{i}u_t = F(t, x, u, u^*)$ are studied. The arbitrary initial value problem for the Black–Scholes model in finance is considered in [15]. Contact transformations and their applications are discussed in Lie [16] as well as [9, 10, 13]. Contact transformations of second-order partial differential equations are used in [17] to obtain *pseudo-invariant* solutions of these second-order partial differential equations. Contact transformations have also been applied to third-order ordinary differential equations to obtain *hidden* transformations [18]. In this paper, we present a method for calculating Lie point transformation generators for evolution-type equations which is both simple and ideal for implementation on a computer algebra package such as MAPLE or MATHEMATICA.

The results of this paper have been applied in [19] to determine Lie point transformations of nonlinear evolution equations and to perform a group classification on a fourth-order nonlinear evolution equation describing the effects of non-uniform surface tension on the spreading of a thin liquid drop. We do not consider the case of more than one dependent variable as it has been proven (see [10]) that systems of equations do not admit contact transformations. An interesting property of evolution equations is that they do admit Lie point and non-trivial Lie–Bäcklund transformations (see [9, 11]), but they do not admit non-trivial contact transformations as we will prove.

In section 2 we briefly discuss the notion of a Lie point transformation and present some well known results for contact transformations which are needed later. In section 3 we prove that evolution equations of the type considered do not admit non-trivial contact transformations. Examples of applications to evolution equations from mathematical physics are given in section 4. Concluding remarks are made in section 5.

2. Preliminaries

We only summarize relevant aspects for the case of two independent variables (time, t, and one space variable, x). The reader is referred to Lie [16] and [9–14].

The set of transformations in (t, x, u) space, namely

$$\bar{t} = \bar{t}(t, x, u, a) \qquad \bar{x} = \bar{x}(t, x, u, a) \qquad \bar{u} = \bar{u}(t, x, u, a) \tag{1}$$

where a is a real parameter, is a one-parameter group of Lie point transformations if it satisfies the group properties. The generator of the group of transformations (1) is given by

$$X = \xi^1(t, x, u)\partial_t + \xi^2(t, x, u)\partial_x + \eta(t, x, u)\partial_u \tag{2}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x, \ldots$. The set of transformations in (t, x, u, u_t, u_x) space, namely

$$\begin{aligned} &\bar{t} = \bar{t}(t, x, u, u_t, u_x, a) \qquad \bar{x} = \bar{x}(t, x, u, u_t, u_x, a) \qquad \bar{u} = \bar{u}(t, x, u, u_x, u_t, a) \\ &\overline{u_t} = \overline{u_t}(t, x, u, u_t, u_x, a) \qquad \overline{u_x} = \overline{u_x}(t, x, u, u_t, u_x, a) \end{aligned} \tag{3}$$

where a is a real parameter, is a one-parameter group of contact transformations if it satisfies the group properties and

$$\overline{u_t} = \frac{\partial \bar{u}}{\partial \bar{t}} \qquad \overline{u_x} = \frac{\partial \bar{u}}{\partial \bar{x}} \tag{4}$$

hold. The generator of a group of contact transformations is

$$\begin{aligned} Y = {} & \xi^1(t, x, u, u_t, u_x)\partial_t + \xi^2(t, x, u, u_t, u_x)\partial_x + \eta(t, x, u, u_x, u_t)\partial_u \\ & + \zeta_1(t, x, u, u_t, u_x)\partial_{u_t} + \zeta_2(t, x, u, u_t, u_x)\partial_{u_x}. \end{aligned} \tag{5}$$

The Lie characteristic function is defined by

$$W = \eta - u_t\xi^1 - u_x\xi^2. \tag{6}$$

The functions ξ^1, ξ^2 and η can be given in terms of W as

$$\xi^1 = -W_{u_t} \qquad \xi^2 = -W_{u_x} \qquad \eta = W - u_t W_{u_t} - u_x W_{u_x}. \tag{7}$$

The formulae for ζ_is can easily be written in terms of W as

$$\zeta_1 = W_t + u_t W_u \qquad \zeta_2 = W_x + u_x W_u. \tag{8}$$

Higher-order prolongations can be calculated from the prolongation formulae

$$\zeta_{i_1 i_2 \ldots i_s} = D_{i_1} \ldots D_{i_s}(W) - W_{u_j} u_{j i_1 \ldots i_s} \qquad s = 1, 2, \ldots \tag{9}$$

with summation on j, where D_i is the operator of total differentiation given by

$$D_i = \partial_{x_i} + u_i \partial_u + u_{ij} \partial_{u_j} + \cdots. \tag{10}$$

If W is linear in the first derivatives u_t and u_x, then the contact transformation generator (5) reduces to an extended Lie point transformation generator of (2).

3. Contact transformations of evolution-type equations

We begin with second-order evolution equations in two independent variables (time, t, and one space variable x). We then consider third-order evolution equations and thereafter naturally extend the result to the general nth-order ($n \geqslant 2$) evolution-type equations in two independent variables. This result is then further extended to the case of an nth-order evolution equation in m independent space variables.

First, we show that contact transformation generators of second-order evolution-type partial differential equations

$$u_t = F(t, x, u, u_x, u_{xx}) \qquad F_{u_{xx}} \neq 0 \tag{11}$$

where

$$F(t, x, u, u_x, u_{xx}) = u_{xx}^k \varphi_k(t, x, u, u_x) \tag{12}$$

with summation over the repeated index k, are just extended Lie point transformation generators given by

$$X = \alpha(t)\partial_t + \beta(t, x, u)\partial_x + \gamma(t, x, u)\partial_u. \tag{13}$$

To determine contact transformations of (11) we solve the determining equation

$$\tilde{X}(u_t - F(t, x, u, u_x, u_{xx}))\big|_{(11)} = 0 \tag{14}$$

where $\tilde{X}$ is the prolongation of the operator (5) in terms of W. Expanding (14) and separating by the mixed derivatives u_{xt}^2 and u_{xt} we obtain

$$u_{xt}^2: \quad W_{u_t u_t} F_{u_{xx}} = 0 \tag{15}$$

$$u_{xt}: \quad (u_{xx} W_{u_t u_x} + u_x W_{u u_t} + W_{x u_t}) F_{u_{xx}} = 0. \tag{16}$$

Since $F_{u_{xx}} \neq 0$, equation (15) implies $W_{u_t u_t} = 0$ and therefore

$$W = u_t C_1(t, x, u, u_x) + C_2(t, x, u, u_x) \tag{17}$$

where C_1 and C_2 are as yet arbitrary functions of t, x, u and u_x. Substituting (17) into (16) and first separating by u_{xx} and then by u_x we find that $C_1 = C_1(t)$ and thus (17) becomes

$$W = u_t C_1(t) + C_2(t, x, u, u_x). \tag{18}$$

If the series (12) is infinite, then one need only consider the truncated series up to some index r (see example 1). In general, most of the coefficients, φ_k, of the series (12) will be zero; e.g. for the heat equation $u_t = u_{xx}$ we observe that $\varphi_1 = 1$ and $\varphi_k = 0$ for $k \neq 1$. Substituting

F into the determining equation and separating by the highest power of u_{xx} namely u_{xx}^{r+1} we obtain

$$u_{xx}^{r+1}\colon\ C_{2_{u_x u_x}} = 0 \tag{19}$$

and therefore

$$C_2 = u_x C_3(t, x, u) + C_4(t, x, u) \tag{20}$$

where C_3 and C_4 are arbitrary functions of t, x and u. Equation (18) is now

$$W = u_t C_1(t) + u_x C_3(t, x, u) + C_4(t, x, u). \tag{21}$$

From (7) we deduce that

$$\xi^1 = -C_1(t) \qquad \xi^2 = -C_3(t, x, u) \qquad \eta = C_4(t, x, u). \tag{22}$$

Hence, the transformation generator corresponding to (21) is given by

$$X = -C_1(t)\partial_t - C_3(t, x, u)\partial_x + C_4(t, x, u)\partial_u. \tag{23}$$

Thus contact transformation generators of (11) are just extended Lie point transformations which have the form (13).

We now show that contact transformation generators of third-order evolution-type partial differential equations

$$u_t = F(t, x, u, u_x, u_{xx}, u_{xxx}) \qquad F_{u_{xxx}} \neq 0 \tag{24}$$

where

$$F(t, x, u, u_x, u_{xx}, u_{xxx}) = u_{xxx}^j u_{xx}^k \varphi_{j,k}(t, x, u, u_x) \tag{25}$$

with summation over the repeated indices j and k, are just extended Lie point transformation generators given by (13). This is easily shown. Indeed, to find contact transformations of (24) we solve the determining equation

$$\tilde{X}(u_t - F(t, x, u, u_x, u_{xx}, u_{xxx}))\big|_{(24)} = 0 \tag{26}$$

where $\tilde{X}$ is the prolongation of the generator (5) in terms of W. Expanding (26) and splitting by the mixed derivatives $u_{xt}u_{xxt}$ and u_{xxt} easily gives

$$u_{xt}u_{xxt}\colon \quad W_{u_t u_t} F_{u_{xxx}} = 0 \tag{27}$$

$$u_{xxt}\colon \quad (u_{xx} W_{u_x u_t} + u_x W_{u u_t} + W_{x u_t}) F_{u_{xxx}} = 0. \tag{28}$$

Since $F_{u_{xxx}} \neq 0$, from (27) we obtain $W_{u_t u_t} = 0$ and therefore

$$W = u_t C_1(t, x, u, u_x) + C_2(t, x, u, u_x) \tag{29}$$

where C_1 and C_2 are as yet arbitrary functions of t, x, u and u_x. Substituting (29) into (28) we obtain $C_1 = C_1(t)$ and therefore (29) becomes

$$W = u_t C_1(t) + C_2(t, x, u, u_x). \tag{30}$$

If the series (25) is infinite, then only a truncation of this series needs to be considered. We substitute F into the determining equation and first separate by the highest power of u_{xxx}. Then we separate the resulting equation by the highest power of u_{xx}. We obtain (19) and hence (23). Thus contact transformation generators of (24) have the form (13).

It is now easy to generalize the above and prove the result for the nth-order case.

Theorem 1. *Contact transformation generators of nth-order evolution-type partial differential equations of the form*

$$u_t = F(t, x, u, u_x, u_{xx}, \ldots, u_{(n)}) \qquad F_{u_{(n)}} \neq 0 \tag{31}$$

where $u_{(n)} = \partial^n u/\partial x^n$ and the function F can be written as a power series in terms of the derivatives $u_{(n)}, u_{(n-1)}, \ldots, u_{xx}$, are just extended Lie point transformation generators given by (13).

Proof. We again solve the determining equation. This time

$$\tilde{X}(u_t - F(t, x, u, u_x, u_{xx}, \ldots, u_{(n)}))\big|_{(31)} = 0 \tag{32}$$

where $\tilde{X}$ is the prolongation of the generator (5) given in terms of W. Expanding (32) and separating by the mixed derivatives $\partial^{n-2}u_{xt}^2/\partial x^{n-2}$ and $\partial^{n-2}u_{xt}/\partial x^{n-2}$ gives

$$\frac{\partial^{n-2}u_{xt}^2}{\partial x^{n-2}}: \quad W_{u_t u_t} F_{u_{(n)}} = 0 \tag{33}$$

$$\frac{\partial^{n-2}u_{xt}}{\partial x^{n-2}}: \quad (u_{xx} W_{u_x u_t} + u_x W_{uu_t} + W_{xu_t}) F_{u_{(n)}} = 0. \tag{34}$$

Since $F_{u_{(n)}} \neq 0$, from (33) we have that $W_{u_t u_t} = 0$ and hence

$$W = u_t C_1(t, x, u, u_x) + C_2(t, x, u, u_x) \tag{35}$$

where C_1 and C_2 are as yet arbitrary functions of t, x, u and u_x. Substituting (35) into (34) we again obtain $C_1 = C_1(t)$ and therefore (35) can be written as

$$W = u_t C_1(t) + C_2(t, x, u, u_x). \tag{36}$$

If the series expansion of F in (31) is infinite, then only a truncation of this series needs to be considered. Substitute F into the determining equation and separate firstly by the highest power of the term $u_{(n)}$. The resulting equation is then separated by the highest power of the term $u_{(n-1)}, \ldots,$ until finally we separate by the highest power of the term u_{xx} to obtain (20) and hence (23). Thus contact transformation generators of (31) have the form (13). □

Hence (22) is the necessary ansatz to use to determine Lie point transformations of (31). When using a computer package such as MAPLE or MATHEMATICA it could be useful to work with (21) instead. Then one need only keep track of W instead of ξ^1, ξ^2 and η.

Corollary 1. *Linear evolution-type equations*

$$u_t = u_{(n)} C_n(t, x) + u_{(n-1)} C_{n-1}(t, x) + \cdots + u_x C_1(t, x) + u C_0(t, x) + \alpha(t, x) \tag{37}$$

where C_j, for $j = 0, \ldots, n$, are arbitrary functions of t and x, do not admit non-trivial contact transformation generators. They collapse to generators of the form (13) which are extended Lie point transformation generators.

Theorem 2. *Contact transformation generators of nth-order evolution-type partial differential equations in m independent variables of the form*

$$u_t = F(t, \boldsymbol{x}, u, u^{(1)}, u^{(2)}, \ldots, u^{(n)}) \tag{38}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ and $u^{(j)}$ is the set of all jth derivatives of u with respect to the space variables $\boldsymbol{x}$ and the function F can be written as a power series in terms of the derivatives $u^{(n)}, u^{(n-1)}, \ldots, u^{(2)}$, are just extended Lie point transformation generators.

Proof. The proof of theorem 2 follows directly from the proof of theorem 1. The required determining equation is given by

$$\tilde{X}(u_t - F(t, \boldsymbol{x}, u, u^{(1)}, u^{(2)}, \dots, u^{(n)})\big|_{(38)} = 0 \tag{39}$$

and the Lie characteristic function W is an arbitrary function of $t, \boldsymbol{x}, u$ and $u^{(1)}$ to be determined. Separating (39) by derivatives of u mixed in time and space, we obtain

$$W = \alpha(t)\, u_t + \beta(t, \boldsymbol{x}, u, u^{(1)}) \tag{40}$$

where α is an arbitrary function of t and β is an arbitrary function of t, $\boldsymbol{x}$, u and $u^{(1)}$. As before, if the power-series expansion for F in (38) is infinite, only a truncation of this series is considered. Substitute F into the determining equation and separate firstly by the highest power of one of the terms $u^{(n)}$. The resulting equation is then separated by the highest power of one of the terms $u^{(n-1)}, \dots,$ until finally we separate by the highest power of one of the terms $u^{(2)}$ to obtain that β is linear in the derivatives $u^{(1)}$, and therefore W is linear in terms of the derivatives $\{u_t, u^{(1)}\}$. Hence, the contact transformation generators of (31) have the form

$$X = \alpha(t)\partial_t + \sum_{i=1}^{m} \beta_i(t, \boldsymbol{x}, u)\partial_{x_i} + \gamma(t, \boldsymbol{x}, u)\partial_u. \tag{41}$$

□

As a consequence of theorem 2, for nth-order evolution-type equations in m independent variables the required ansatz for determining Lie point transformations in terms of the Lie characteristic function is given by

$$W = \alpha(t)\, u_t + \sum_{i=1}^{m} C_i(t, \boldsymbol{x}, u)\, u_{x_i} \tag{42}$$

where α is an arbitrary function of t and C_i is an arbitrary function of t, x and u. Alternatively, using the fact that

$$\xi^j = -W_{u_j} \qquad \eta = W - u_i W_{u_i} \qquad j = 1, \dots, m \tag{43}$$

with summation over the repeated index i, equation (43) gives the required ansatz for determining Lie point transformations from the standard approach. When using a computer algebra package like MAPLE or MATHEMATICA, it could be more convenient to work with the Lie characteristic function W, as one need only keep track of W.

4. Examples

Examples verifying the results of theorem 1 are listed under the subheading example 1. Examples showing applications of the results from theorems 1 and 2 for determining Lie point transformations are given under the subheading example 2.

4.1. Example 1

In this example we verify that for an infinite series, the required ansatz for the Lie characteristic function to determine Lie point transformations is given by (21).

Consider the equation

$$u_t = \sin u_{xx}. \tag{44}$$

The series expansion of $\sin u_{xx}$ is given by

$$\sin u_{xx} = u_{xx} - \tfrac{1}{6}u_{xx}^3 + \tfrac{1}{120}u_{xx}^5 + \cdots. \tag{45}$$

Using (45), equation (44) is

$$u_t = u_{xx} - \tfrac{1}{6}u_{xx}^3 + \tfrac{1}{120}u_{xx}^5 + \cdots. \tag{46}$$

The determining equation (14) where $F = u_{xx} - \frac{1}{6}u_{xx}^3 + \frac{1}{120}u_{xx}^5 + \cdots$ and (46) holds becomes

$$\begin{aligned} -W_t - \left(u_{xx} - \tfrac{1}{6}u_{xx}^3 + \tfrac{1}{120}u_{xx}^5 + \cdots\right)W_u + (&u_{xx}^2 W_{u_x u_x} + 2u_{xt}u_{xx}W_{u_t u_x} + u_{xt}^2 W_{u_t u_t} + u_{xx}W_u \\ &+2u_x u_{xx}W_{uu_x} + 2u_x u_{xt}W_{uu_t} + u_x^2 W_{uu} + 2u_{xx}W_{xu_x} \\ &+2u_{xt}W_{xu_t} + 2u_x W_{xu} + W_{xx})\left(1 - \tfrac{1}{2}u_{xx}^2 + \tfrac{1}{24}u_{xx}^4 + \cdots\right) = 0. \end{aligned} \tag{47}$$

We can separate (47) by the mixed derivatives u_{tx}^2 and u_{tx} to obtain (18) independently of the truncation of the series. Substitute (18) into (47). Truncating the series after three terms and separating by the highest power of the derivative u_{xx} namely u_{xx}^6 we obtain (19) and hence (21).

Consider Burgers' equation

$$u_t = uu_x + u_{xx}. \tag{48}$$

The determining equation (14) with $F = uu_x + u_{xx}$ and $u_t = uu_x + u_{xx}$ reduces to

$$\begin{aligned} (W - (uu_x + u_{xx})W_{u_t} - u_x W_{u_x})\, u_x - W_t - (uu_x + u_{xx})W_u + (W_x + u_x W_u)\, u& \\ +\big(u_{xx}^2 W_{u_x u_x} + 2u_{xt}u_{xx}W_{u_t u_x} + u_{xt}^2 W_{u_t u_t} + u_{xx}W_u + 2u_x u_{xx}W_{uu_x}& \\ +2u_x u_{xt}W_{uu_t} + u_x^2 W_{uu} + 2u_{xx}W_{xu_x} + 2u_{xt}W_{xu_t} + 2u_x W_{xu} + W_{xx}\big) = 0.& \end{aligned} \tag{49}$$

Separate (49) by the mixed derivatives u_{xt}^2 and u_{xt} to obtain

$$W_{u_t u_t} = 0 \qquad u_{xx}W_{u_t u_x} + u_x W_{uu_t} + W_{xu_t} = 0 \tag{50}$$

and hence (18). Substituting (18) into (49) and separating by u_{xx}^2 we obtain (19) and hence (21).

Consider the general Hopf equation

$$u_t = -uu_x + (k(u)u_x)_x. \tag{51}$$

The determining equation (14) with $F = -uu_x + (k(u)u_x)_x$ and $u_t = -uu_x + (k(u)u_x)_x$ is given by

$$\begin{aligned} (W - (-uu_x + (k(u)u_x)_x)W_{u_t} - u_x W_{u_x})(-u_x + k''(u)u_x^2 + k'(u)\, u_{xx})& \\ -W_t - (-uu_x + k'(u)u_x^2 + k(u)u_{xx})W_u + (W_x + u_x W_u)(-u + 2k'(u)u_x)& \\ +\big(u_{xx}^2 W_{u_x u_x} + 2u_{xt}u_{xx}W_{u_t u_x} + u_{xt}^2 W_{u_t u_t} + u_{xx}W_u + 2u_x u_{xx}W_{uu_x}& \\ +2u_x u_{xt}W_{uu_t} + u_x^2 W_{uu} + 2u_{xx}W_{xu_x} + 2u_{xt}W_{xu_t} + 2u_x W_{xu} + W_{xx}\big)k(u) = 0.& \end{aligned} \tag{52}$$

As before separate (52) by the mixed derivatives u_{xt}^2 and u_{xt} to obtain (18). Substituting (18) into (52) and separating by u_{xx}^2 we obtain (19) and hence (21).

Consider the equation

$$u_t = u_{xx}^{1/3}. \tag{53}$$

Substituting (53) with $F = u_{xx}^{1/3}$ into the determining equation (14) we obtain

$$-W_t - u_{xx}^{1/3} W_u + \big(u_{xx}^2 W_{u_x u_x} + 2u_{xt}u_{xx}W_{u_t u_x} + u_{xt}^2 W_{u_t u_t} + u_{xx}W_u + 2u_x u_{xx} W_{uu_x} \\ +2u_x u_{xt} W_{uu_t} + u_x^2 W_{uu} + 2u_{xx}W_{xu_x} + 2u_{xt}W_{xu_t} + 2u_x W_{xu} + W_{xx}\big)\tfrac{1}{3}u_{xx}^{-2/3} = 0. \quad (54)$$

The result follows as from the previous examples.

Note that we have chosen to verify the results from theorems 1 using second-order examples due to the simplicity of their determining equations. For third and higher-order equations the determining equations are longer. However, the result still holds.

4.2. Example 2

In this example we use the ansatz

$$W = C_1(t)\, u_t + C_2(t, x, u)\, u_x + C_3(t, x, u) \quad (55)$$

to obtain Lie point transformations of the heat equation

$$u_t = u_{xx}. \quad (56)$$

The determining equation (14) with $F = u_{xx}$ and where (56) holds reduces to

$$-W_t - u_{xx} W_u + \big(u_{xx}^2 W_{u_x u_x} + 2u_{xt}u_{xx}W_{u_t u_x} + u_{xt}^2 W_{u_t u_t} + u_{xx}W_u + 2u_x u_{xx} W_{uu_x} \\ +2u_x u_{xt} W_{uu_t} + u_x^2 W_{uu} + 2u_{xx}W_{xu_x} + 2u_{xt}W_{xu_t} + 2u_x W_{xu} + W_{xx}\big) = 0. \quad (57)$$

Substituting the required ansatz (55) into (57) and separating by the remaining derivatives of u we obtain

$$\begin{array}{lll} C_{3_t} - C_{3_{xx}} = 0 & C_{2_t} - 2C_{3_{xu}} - C_{2_{xx}} = 0 & C_{3_{uu}} + 2C_{2_{xu}} = 0 \\ C_{1_t} - 2C_{2_x} = 0 & C_{2_u} = 0. & \end{array} \quad (58)$$

Solving the system (58) we obtain

$$W = \left(\tfrac{1}{4}t\alpha_1 + \tfrac{1}{8}x^2\alpha_1 + \tfrac{1}{2}\alpha_4 + \alpha_6\right) u + \beta(t, x) + \left(\tfrac{1}{2}xt\alpha_1 + \tfrac{1}{2}x\alpha_2 + t\alpha_4 + \alpha_5\right) u_x \\ +\left(\tfrac{1}{2}t^2\alpha_1 + t\alpha_2 + \alpha_3\right) u_t \quad (59)$$

where the α_is are constants and the function $\beta(t, x)$ satisfies $\beta_t - \beta_{xx} = 0$. Substituting (59) into (7) we obtain the Lie point transformation generators

$$\begin{array}{lllll} Y_1 = \partial_t & Y_2 = \partial_x & Y_3 = u\partial_u & Y_4 = 2t\partial_t + x\partial_x & Y_5 = 2t\partial_x - xu\partial_u \\ Y_6 = 4t^2\partial_t + 4xt\partial_x - (2t + x^2)u\partial_u & Y_\beta = \beta(t, x)\partial_u & & & \end{array} \quad (60)$$

which were determined by Lie [20].

To determine Lie point transformations of the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy} \quad (61)$$

we impose the ansatz

$$W = C_1(t)\, u_t + C_2(t, x, y, u)\, u_x + C_3(t, x, y, u)\, u_y + C_4(t, x, y, u) \quad (62)$$

on the determining equation obtained from (39) when $F = u_{xx} + u_{yy}$ and (61) holds. Separating the resulting determining equation by the remaining derivatives of u we obtain the system

$$\begin{aligned} &C_{4_t} - C_{4_{xx}} - C_{4_{tt}} = 0 \qquad C_{3_t} - 2C_{4_{yu}} - C_{3_{xx}} - C_{3_{yy}} = 0 \qquad C_{4_{uu}} + 2C_{3_{yu}} = 0 \\ &C_{2_t} - C_{2_{xx}} - C_{2_{yy}} - 2C_{4_{xu}} = 0 \qquad C_{2_{yu}} + 2C_{3_{xu}} = 0 \qquad C_{4_{uu}} + 2C_{2_{xu}} = 0 \\ &C_{3_{uu}} = 0, 2C_{2_y} + 2C_{3_x} = 0 \qquad C_{2_u} = 0 \qquad C_{3_u} = 0 \\ &C_{1_t} - 2C_{3_y} = 0 \qquad C_{1_t} - 2C_{2_x} = 0 \end{aligned} \tag{63}$$

which can easily be solved to give

$$\begin{aligned} W = {} & \tfrac{1}{8}\left(4t\alpha_1 + x^2\alpha_1 + y^2\alpha_1 + 4x\alpha_4 + 4y\alpha_6 + 8\alpha_8\right) u + \beta(t, x, y) \\ & + \left(\tfrac{1}{2}yt\alpha_1 + \tfrac{1}{2}\alpha_2 + t\alpha_6 + \alpha_7 - x\alpha_9\right) u_y + \left(\tfrac{1}{2}xt\alpha_1 + \tfrac{1}{2}x\alpha_2 + t\alpha_4 + \alpha_5 + y\alpha_9\right) u_x \\ & + \left(\tfrac{1}{2}t^2\alpha_1 + t\alpha_2 + \alpha_3\right) u_t \end{aligned} \tag{64}$$

where the α_is are arbitrary constants and β satisfies the equation $\beta_t - \beta_{xx} - \beta_{yy} = 0$. Substituting (64) into (43) we obtain the Lie point transformation generators

$$\begin{aligned} &Y_1 = \partial_t \qquad Y_2 = \partial_x \qquad Y_3 = \partial_y \qquad Y_4 = y\partial_x - x\partial_y \qquad Y_5 = u\partial_u \\ &Y_6 = 2t\partial_x - xu\partial_u \qquad Y_7 = 2t\partial_y + yu\partial_u \qquad Y_8 = 2t\partial_t + x\partial_x + y\partial_y \\ &Y_9 = 4t^2\partial_t + 4xt\partial_x + 4yt\partial_y - (4t + x^2 + y^2)u\partial_u \qquad Y_\beta = \beta(t, x, y)\partial_u. \end{aligned} \tag{65}$$

To determine Lie point transformations of the nonlinear evolution equation

$$u_t = u_{xy}u_{xxy} + u^2u_xu_y \tag{66}$$

we impose the ansatz (62) on the determining equation obtained from (39) when $F = u_{xy}u_{xxy} + u^2u_xu_y$ and (66) holds. Separating the determining equation by the remaining derivatives of u and solving we obtain

$$W = \alpha_4 u + \left(\alpha_3 - \tfrac{1}{4}y(\alpha_4 + 7\alpha_5)\right) u_y + (x\alpha_5 + \alpha_2)\, u_x + \left(\tfrac{1}{4}t(11\alpha_4 - 3\alpha_5) + \alpha_1\right) u_t \tag{67}$$

where the α_is are constant, and hence from (43), we obtain the Lie point symmetry generators

$$\begin{aligned} &Y_1 = \partial_t \qquad Y_2 = \partial_x \qquad Y_3 = \partial_y \\ &Y_4 = -11t\partial_t + y\partial_y + 4u\partial_u \qquad Y_5 = 3t\partial_t - 4x\partial_x + 7y\partial_y. \end{aligned} \tag{68}$$

5. Concluding remarks

We have shown that evolution equations of the type considered do not admit a non-trivial one-parameter group of contact transformations. Consequently, we have shown that the required ansatz to determine Lie point transformations of evolution-type equations from the contact transformation approach is given by (42). The contact transformation approach is useful for determining Lie point transformations of evolution equations when using a computer algebra package such as MAPLE or MATHEMATICA as one needs to keep track of W only instead of the ξ^is and η.

Exact solutions for evolution equations can be determined from their classical [10–14] and non-classical [21] symmetries. Notwithstanding, Feinsilver and co-workers [22–25] have shown that, in particular, polynomial solutions of evolution equations can be given in terms of Appell systems (see equation (4.3) on p 262 [23]). Further work needs to be done to establish a possible link between the Lie method as used here and the method discussed in [22–25].

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